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The Differential Transform Method for Solving the Burgers Equation

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Abstract: **The Burgers equation is a fundamental partial differential equation (PDE) in applied mathematics, mathematical physics, and engineering, describing a variety of nonlinear wave phenomena such as fluid dynamics, gas dynamics, and traffic flow. Its nonlinear nature makes it a challenging yet fascinating problem to solve analytically or numerically. Among the various methods developed to tackle the Burgers equation, the Differential Transform Method (DTM) has emerged as a powerful semi-analytical technique due to its simplicity, computational efficiency, and ability to handle nonlinear PDEs. This article provides an in-depth exploration of the DTM and its application to solving the Burgers equation, covering its theoretical foundations, implementation, advantages, limitations, and illustrative examples.**

Keywords: **partial differential equation (PDE), Differential Transform Method (DTM), Burgers equation.**

1. INTRODUCTION TO THE BURGERS EQUATION

The Burgers equation, named after Dutch physicist Johannes Martinus Burgers, is a nonlinear PDE that combines both convection and diffusion effects. In its one-dimensional form, it is expressed as:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = v\frac{\partial^2 u}{\partial x^2}$$

where $u(x,t)$ represents the dependent variable, t is time, x is the spatial coordinate, and $v > 0$ is the kinematic viscosity, which governs the diffusion term. The term $u\frac{\partial u}{\partial x}$ introduces nonlinearity, making the equation a simplified model for studying nonlinear wave propagation and shock formation.

The Burgers equation serves as a prototype for understanding more complex PDEs, such as the Navier-Stokes equations, and is widely used in fields like fluid mechanics, acoustics, and traffic flow modeling. Exact solutions to the Burgers equation are possible in specific cases, but for general initial and boundary conditions, numerical or semi-analytical methods are often required.

The Differential Transform Method (DTM) is one such semi-analytical approach that has gained popularity for solving nonlinear PDEs like the Burgers equation. DTM transforms the PDE into a set of algebraic equations, which are easier to solve iteratively, providing series solutions that approximate the exact solution with high accuracy.

2. THE DIFFERENTIAL TRANSFORM METHOD: THEORETICAL FOUNDATIONS

The DTM is a semi-analytical method based on the Taylor series expansion, originally introduced by J.K. Zhou in 1986 for solving linear and nonlinear differential equations in electrical circuit analysis. It has since been extended to solve ordinary differential equations (ODEs), PDEs, and integral equations. The method transforms a differential equation and its initial or boundary conditions into a set of algebraic recurrence relations in the transform domain, which can be solved to obtain the coefficients of a power series solution.

One-Dimensional Differential Transform

For a function $u(x)$, the one-dimensional differential transform is defined as:

$$U(k) = \frac{1}{k!}\left[\frac{d^k u(x)}{dx^k}\right]_{x=x_0}$$

where $U(k)$ is the transformed function (or differential transform) of $u(x)$, k is a non-negative integer representing the order of the transform, and $x_0$is the point about which the Taylor series is expanded (often chosen as $x_0 = 0$). The inverse differential transform is given by:

$$u(x) = \sum_{k=0}^{\infty} U(k)(x - x_0)^k$$

This inverse transform reconstructs the original function as a power series, which approximates $u(x)$ within a certain radius of convergence.

Two-Dimensional Differential Transform

For PDEs like the Burgers equation, which involve two independent variables, the two-dimensional DTM is used. For a function $u(x,t)$, the two-dimensional differential transform is defined as:

$$U(k,h) = \frac{1}{k!\,h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial t^h} u(x,t)\right]_{(x_0,t_0)}$$

where $U(k,h)$ is the transformed function, and k and h are the orders of differentiation with respect to x and t, respectively. The inverse transform is:

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)(x - x_0)^k (t - t_0)^h$$

Typically, $(x_0, t_0) = (0,0)$ is chosen for simplicity unless otherwise specified.

Key Properties of DTM

The DTM relies on a set of operational rules to transform differential equations into algebraic equations. Some fundamental properties include:

1. **Linearity:** If $u(x,t) = \alpha v(x,t) + \beta w(x,t)$, then

$U(k,h) = \alpha V(k,h) + \beta W(k,h)$.

2. **Differentiation:** If $w(x,t) = \frac{\partial u(x,t)}{\partial x}$ then $W(k,h) = (k+1)U(k+1,h)$**.**

3. **Product Rule:** If $w(x,t) = u(x,t)v(x,t)$ **,** then

$$W(k,h) = \sum_{m=0}^{k}\sum_{n=0}^{h} U(m,n)V(k-m,h-n).$$

Initial Conditions: Initial conditions are incorporated directly into the transform, $u(0,0) = a$ implies $U(0,0) = a$.

These properties allow the DTM to handle nonlinear terms, such as the convective term $u\frac{\partial u}{\partial x}$ in the Burgers equation, by transforming them into algebraic convolutions.

Applying DTM to the Burgers Equation

To illustrate the application of DTM to the Burgers equation, consider the one-dimensional Burgers equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

with initial condition: $u(x,0) = f(x)$ and appropriate boundary conditions, such as $u(0,t) = g(t)$ and $u(L,t) = h(t)$, where L is the spatial domain length.

Step 1: Transform the Equation

Using the two-dimensional DTM, we transform each term of the Burgers equation:

1. **Time Derivative:**

$$\frac{\partial u}{\partial t} \rightarrow (h+1)U(k,h+1)$$

2. **Second Spatial Derivative:**

$$\frac{\partial^2 u}{\partial x^2} \rightarrow (k+1)(k+2)U(k+2,h)$$

3. **Nonlinear Term $u\frac{\partial u}{\partial x}$:**

Let $v = \frac{\partial u}{\partial x}$, so $V(k,h) = (k+1)\,U(k+1,h)$. The product $w = uv = \frac{\partial u}{\partial x}$ transforms as:

$$\sum_{n=0}^{h} U(m,n)(k-m+1)U(k-m,h-n)$$

Applying DTM to the entire equation, we obtain the recurrence relation:

$$(h+1)U(k,h+1) + \sum_{m=0}^{k}\sum_{n=0}^{h} U(m,n)(k-m+1)U(k-m+1,h-n) \quad = v(k+1)(k+2)U(k+2,h)$$

Step 2: Transform Initial and Boundary Conditions

The initial condition $u(x,0) = f(x)$ is transformed using the one-dimensional DTM:

$$U(k,0) = \frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=0}$$

Boundary conditions, if provided, are similarly transformed. For example, $u(0,t) = g(t)$ gives:

$$U(0,h) = \frac{1}{h!}\left[\frac{d^h g(t)}{dt^h}\right]_{t=0}$$

Step 3: Solve the Recurrence Relation

Starting with $U(k,0)$ from the initial condition, the recurrence relation is solved iteratively for $U(k,h)$ for $h = 0,1,2,\ldots$ This process generates the coefficients of the series solution. For practical computations, the series is truncated at a finite number of terms, say N, to obtain an approximate solution:

$$u(x,t) \approx \sum_{k=0}^{N}\sum_{h=0}^{N} U(k,h)x^k t^h$$

Step 4: Reconstruct the Solution

The inverse transform is used to construct the approximate solution $u(x,t)$. The accuracy of the solution depends on the number of terms included and the radius of convergence of the series.

Illustrative Example

Consider the Burgers equation with the initial condition:

$u(x,0) = \sin(\pi x),\ 0 \le x \le 1$,

and boundary conditions: $u(0,t) = u(1,t) = 0,\ t \ge 0$,

with viscosity $v = 0.01$.

Step 1: Transform the Initial Condition

The initial condition $u(x,0) = \sin(\pi x)$ is transformed:

$$U(k,0) = \frac{1}{k!}\left[\frac{d^k sin(\pi x)}{dx^k}\right]_{x=0}$$

Compute the derivatives:

$\sin(\pi x) \to \cos(\pi x) \to -\pi \sin(\pi x) \to \cdots$, etc.

At $x = 0$:

- $k = 0$: $sin(0) = 0$, $so\ U(0,0) = 0$.
- $k = 1$: $\pi cos(0) = \pi$, $so\ U(1,0) = \pi$.
- $k = 2$: $-\pi^2 sin(0) = 0$, $so\ U(2,0) = 0$.
- $k = 3$: $-\pi^3 cos(0) = -\pi^3$, $so\ U(3,0) = -\frac{\pi^3}{3!} = -\frac{\pi^3}{6}$

Thus, $U(k,0)$ is nonzero for odd k.

Step 2: Apply the Recurrence Relation

Using the recurrence relation, compute $U(k,h)$ for $h \geq 1$. For simplicity, truncate at a small number of terms and solve for the coefficients. This process is computationally intensive for higher orders, but symbolic computation tools like MATLAB or Mathematica can automate it.

Step 3: Construct the Solution

The series solution is constructed by summing the terms. For a small number of terms, the solution approximates the behavior of the exact solution, which can be compared to numerical methods like finite difference or exact solutions via the Cole-Hopf transformation.

Advantages of DTM for the Burgers Equation

1. **Simplicity:** DTM transforms the PDE into algebraic equations, which are easier to solve than differential equations.
2. **Handling Nonlinearity:** The method effectively handles nonlinear terms through convolution operations.
3. **Semi-Analytical Nature:** DTM provides a series solution that can be analyzed symbolically, offering insights into the solution's behavior.
4. **Flexibility:** It can be applied to various initial and boundary conditions without significant modifications.

Limitations of DTM

1. **Convergence Issues:** The series solution may have a limited radius of convergence, requiring truncation or alternative methods for long-time behavior.
2. **Computational Complexity:** For high-order terms, the recurrence relations become complex, necessitating computational tools.
3. **Boundary Conditions:** Incorporating complex boundary conditions can be challenging and may require additional techniques like the differential transform at boundaries.

3. COMPARISON WITH OTHER METHODS

Compared to numerical methods like finite difference or finite element methods, DTM offers a semi-analytical solution that avoids discretization errors but may be less efficient for large domains or long times. Compared to exact methods like the Cole-Hopf transformation, DTM is more versatile for general conditions but less precise for specific cases where exact solutions exist.

4. CONCLUSION

The Differential Transform Method is a robust and versatile tool for solving the Burgers equation, offering a balance between analytical insight and computational feasibility. By transforming the nonlinear PDE into a set of algebraic recurrence relations, DTM provides a series solution that approximates the exact solution with high accuracy for small times and domains. Its ability to handle nonlinearity and flexibility with initial conditions make it an attractive choice for researchers and engineers. However, care must be taken to address convergence issues and computational complexity for practical applications. With advancements in symbolic computation, DTM continues to be a valuable method for tackling nonlinear PDEs like the Burgers equation, contributing to the broader understanding of nonlinear wave phenomena.

REFERENCES

[1] Benton, E. R., & Platzman, G. W. (1972). A table of solutions of the one-dimensional Burgers equation. *Quarterly of Applied Mathematics, 30*(2), 195–212. https://doi.org/10.1090/qam/307279

[2] Burgers, J. M. (1948). A mathematical model illustrating the theory of turbulence. *Advances in Applied Mechanics, 1*, 171–199. https://doi.org/10.1016/S0065-2156(08)70100-5

[3] Cole, J. D. (1951). On a quasi-linear parabolic equation occurring in aerodynamics. *Quarterly of Applied Mathematics, 9*(3), 225–236. https://doi.org/10.1090/qam/42891

[4] Hopf, E. (1950). The partial differential equation \(u_t + u u_x = \mu u_{xx} \). *Communications on Pure and Applied Mathematics, 3*(3), 201–230. https://doi.org/10.1002/cpa.3160030302

[5] Jang, T. S., Baek, H., & Park, J. (2010). A new method for solving the Burgers equation via the differential transform method. *Applied Mathematics and Computation, 217*(8), 3899–3906. https://doi.org/10.1016/j.amc.2010.09.055

[6] Zhou, J. K. (1986). *Differential transformation and its applications for electrical circuits*. Huazhong University Press.

[7] Sarı, M., & Gürarslan, G. (2009). A hybrid differential transform-finite difference method for solving Burgers' equation. *Mathematical and Computer Modelling, 50*(7-8), 1125–1133. https://doi.org/10.1016/j.mcm.2009.05.028

[8] Whitham, G. B. (1974). *Linear and nonlinear waves*. John Wiley & Sons.